N-Fractional Calculus of Some Irrational Functions (Study on Non-Analytic and Univalent Functions and Applications)

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N- Fractional Calculus of Some Irrational Functions

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Abstract

In this article N- fractional calculus of the functions (which have multiple root signs)

$$f(z)=\sqrt{\sqrt{\sqrt{z-b}-c}-d}$$

are discussed .

Theorem 1. *We have*

(i)
$$\left(f(z)\right)_\gamma = e^{-i\pi\gamma}(z-b)^{(1/8)-\gamma}\sum_{m,k=0}^{\infty}\frac{[-\frac{1}{2}]_m[\frac{m}{2}-\frac{1}{4}]_k\Gamma(\frac{k}{2}-\frac{1}{8}+\frac{m}{4}+\gamma)}{m!\cdot k!\;\Gamma(\frac{k}{2}-\frac{1}{8}+\frac{m}{4})}$$

$$\times\left(\frac{c}{\sqrt{z-b}}\right)^k\left(\frac{d}{\sqrt[4]{z-b}}\right)^m, \qquad \left(\left|\frac{\Gamma(\frac{k}{2}-\frac{1}{8}+\frac{m}{4}+\gamma)}{\Gamma(\frac{k}{2}-\frac{1}{8}+\frac{m}{4})}\right|<\infty\right)$$

and

(ii)
$$\left(f(z)\right)_n = (-1)^n(z-b)^{(1/8)-n}\sum_{m,k=0}^{\infty}\frac{[-\frac{1}{2}]_m[\frac{m}{2}-\frac{1}{4}]_k[\frac{k}{2}-\frac{1}{8}+\frac{m}{4}]_n}{m!\cdot k!}$$

$$\times\left(\frac{c}{\sqrt{z-b}}\right)^k\left(\frac{d}{\sqrt[4]{z-b}}\right)^m, \qquad (n\in Z_0^+)$$

where

$$\left|\frac{c}{\sqrt{z-b}}\right|<1\;,\quad\left|\frac{d}{\sqrt[4]{z-b}}\right|<1\;.$$

§ 0. Introduction (Definition of Fractional Calculus)

(I) Definition. (by K. Nishimoto) ([1] Vol. 1)

Let $D = \{D_-, D_+\}$, $C = \{C_-, C_+\}$,

C_- be a curve along the cut joining two points z and $-\infty + i\,\mathrm{Im}(z)$,

C_+ be a curve along the cut joining two points z and $\infty + i\,\mathrm{Im}(z)$,

D_- be a domain surrounded by C_- , D_+ be a domain surrounded by C_+ .

(Here D contains the points over the curve C).

Moreover, let $f = f(z)$ be a regular function in $D (z \in D)$,

$$f_\nu = (f)_\nu = {}_C(f)_\nu = \frac{\Gamma(\nu+1)}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z)^{\nu+1}}\,d\zeta \qquad (\nu \notin Z^-), \tag{1}$$

$$(f)_{-m} = \lim_{\nu \to -m} (f)_\nu \qquad (m \in Z^+), \tag{2}$$

where $\quad -\pi \le \arg(\zeta - z) \le \pi$ for C_- , $\quad 0 \le \arg(\zeta - z) \le 2\pi$ for C_+ ,

$\zeta \ne z$, $z \in C$, $\nu \in R$, Γ ; Gamma function,

then $(f)_\nu$ is the fractional differintegration of arbitrary order ν (derivatives of order ν for $\nu > 0$, and integrals of order $-\nu$ for $\nu < 0$), with respect to z , of the function f , if $|(f)_\nu| < \infty$.

(I I) On the fractional calculus operator N^ν [3]

Theorem A. *Let fractional calculus operator (Nishimoto's Operator) N^ν be*

$$N^\nu = \left(\frac{\Gamma(\nu+1)}{2\pi i}\int_C \frac{d\zeta}{(\zeta - z)^{\nu+1}}\right) \qquad (\nu \notin Z^-), \quad [\,\textit{Refer to}\,(1)\,] \tag{3}$$

with

$$N^{-m} = \lim_{\nu \to -m} N^\nu \qquad (m \in Z^+), \tag{4}$$

and define the binary operation $\circ$ *as*

$$N^\beta \circ N^\alpha f = N^\beta N^\alpha f = N^\beta (N^\alpha f) \qquad (\alpha, \beta \in R), \tag{5}$$

then the set

$$\{N^\nu\} = \{N^\nu \mid \nu \in R\} \tag{6}$$

is an Abelian product group (having continuous index ν) which has the inverse transform operator $(N^\nu)^{-1} = N^{-\nu}$ to the fractional calculus operator N^ν , for the function f such that $f \in F = \{f\,;\, 0 \ne |f_\nu| < \infty, \nu \in R\}$, where $f = f(z)$ and $z \in C$.
(vis. $-\infty < \nu < \infty$).

(For our convenience, we call $N^{\beta} \circ N^{\alpha}$ as product of N^{β} and N^{α}.)

Theorem B. " F.O.G. $\{N^{\nu}\}$ " *is an* " *Action product group which has continuous index* ν " *for the set of* F . (*F.O.G.* ; Fractional calculus operator group) [3]

(III) Lemma. We have [1]

(i) $$((z-c)^{b})_{\alpha} = e^{-i\pi\alpha}\frac{\Gamma(\alpha-b)}{\Gamma(-b)}(z-c)^{b-\alpha} \qquad \left(\left|\frac{\Gamma(\alpha-b)}{\Gamma(-b)}\right|<\infty\right), \tag{7}$$

(ii) $$(\log(z-c))_{\alpha} = -e^{-i\pi\alpha}\Gamma(\alpha)(z-c)^{-\alpha} \qquad (|\Gamma(\alpha)|<\infty), \tag{8}$$

(iii) $$((z-c)^{-\alpha})_{-\alpha} = -e^{i\pi\alpha}\frac{1}{\Gamma(\alpha)}\log(z-c) \qquad (|\Gamma(\alpha)|<\infty), \tag{9}$$

where $z-c \neq 0$ for (i) and $z-c \neq 0, 1$ for (i i), (i i i) ,

§ 1. Preliminary

[I] The theorem below is reported by K. Nishimoto already (cf. J. Frac. Calc. Vol. 29, May (2006), p.37). [12]

Theorem D. *We have*

(i) $$(((z-b)^{\beta}-c)^{\alpha})_{\gamma} = e^{-i\pi\gamma}(z-b)^{\alpha\beta-\gamma}$$

$$\times\sum_{k=0}^{\infty}\frac{[-\alpha]_k\Gamma(\beta k-\alpha\beta+\gamma)}{k!\,\Gamma(\beta k-\alpha\beta)}\left(\frac{c}{(z-b)^{\beta}}\right)^{k} \qquad \left(\left|\frac{\Gamma(\beta k-\alpha\beta+\gamma)}{\Gamma(\beta k-\alpha\beta)}\right|<\infty\right) \tag{1}$$

and

(ii) $$(((z-b)^{\beta}-c)^{\alpha})_{n} = (-1)^{n}(z-b)^{\alpha\beta-n}$$

$$\times\sum_{k=0}^{\infty}\frac{[-\alpha]_k[\beta k-\alpha\beta]_n}{k!}\left(\frac{c}{(z-b)^{\beta}}\right)^{k} \qquad (n\in Z_0^{+}) \tag{2}$$

where

$$|c/(z-b)^{\beta}|<1,$$

and

$$[\lambda]_k = \lambda(\lambda+1)\cdots(\lambda+k-1) = \Gamma(\lambda+k)/\Gamma(\lambda) \quad \textit{with} \quad [\lambda]_0 = 1.$$

(Notation of Pochhammer).

[I I] The theorem below is reported by K. Nishimoto already (cf. J. Frac. Calc. Vol. 31, May (2007), p.13). [13]

Theorem E. *We have*

(i) $$((((z-b)^{\beta}-c)^{\alpha}-d)^{\delta})_{\gamma}=e^{-i\pi\gamma}(z-b)^{\alpha\beta\delta-\gamma}$$

$$\times\sum_{m,k=0}^{\infty}\frac{[-\delta]_m[-\alpha(\delta-m)]_k\Gamma(\beta k-\alpha\beta(\delta-m)+\gamma)}{m!\cdot k!\,\Gamma(\beta k-\alpha\beta(\delta-m))}\left(\frac{c}{(z-b)^{\beta}}\right)^{k}\left(\frac{d}{(z-b)^{\alpha\beta}}\right)^{m} \tag{3}$$

$$\left(\left|\frac{\Gamma(\beta k-\alpha\beta(\delta-m)+\gamma)}{\Gamma(\beta k-\alpha\beta(\delta-m))}\right|<\infty\right)$$

and

(ii) $$((((z-b)^{\beta}-c)^{\alpha}-d)^{\delta})_{n}=(-1)^{n}(z-b)^{\alpha\beta\delta-n}$$

$$\times\sum_{m,k=0}^{\infty}\frac{[-\delta]_m[-\alpha(\delta-m)]_k[\beta k-\alpha\beta(\delta-m)]_n}{m!\cdot k!}\left(\frac{c}{(z-b)^{\beta}}\right)^{k}\left(\frac{d}{(z-b)^{\alpha\beta}}\right)^{m} \tag{4}$$

$$(n\in Z_0^{+})$$

where

$$((z-b)^{\beta}-c)^{\alpha}-d\neq 0,\quad |c/(z-b)^{\beta}|<1,\quad |d/(z-b)^{\alpha\beta}|<1.$$

§ 2. N- Fractional Calculus of Functions

$$\sqrt{\sqrt{\sqrt{z-b}-c}-d}$$

Theorem 1. *We have*

(i) $$\left(\sqrt{\sqrt{\sqrt{z-b}-c}-d}\right)_{\gamma}=e^{-i\pi\gamma}(z-b)^{(1/8)-\gamma}$$

$$\times\sum_{m,k=0}^{\infty}\frac{[-\frac{1}{2}]_m[\frac{m}{2}-\frac{1}{4}]_k\Gamma(\frac{k}{2}-\frac{1}{8}+\frac{m}{4}+\gamma)}{m!\cdot k!\,\Gamma(\frac{k}{2}-\frac{1}{8}+\frac{m}{4})}\left(\frac{c}{\sqrt{z-b}}\right)^{k}\left(\frac{d}{\sqrt[4]{z-b}}\right)^{m} \tag{1}$$

$$\left(\left|\frac{\Gamma(\frac{k}{2}-\frac{1}{8}+\frac{m}{4}+\gamma)}{\Gamma(\frac{k}{2}-\frac{1}{8}+\frac{m}{4})}\right|<\infty\right)$$

and

(ii) $$\left(\sqrt{\sqrt{\sqrt{z-b}-c}-d}\right)_n = (-1)^n (z-b)^{(1/8)-n}$$

$$\times \sum_{m,k=0}^{\infty} \frac{[-\frac{1}{2}]_m [\frac{m}{2}-\frac{1}{4}]_k [\frac{k}{2}-\frac{1}{8}+\frac{m}{4}]_n}{m! \cdot k!} \left(\frac{c}{\sqrt{z-b}}\right)^k \left(\frac{d}{\sqrt[4]{z-b}}\right)^m \tag{2}$$

$$(n \in \mathbf{Z}_0^+)$$

where

$$\left|\frac{c}{\sqrt{z-b}}\right| < 1 \quad , \quad \left|\frac{d}{\sqrt[4]{z-b}}\right| < 1 \ .$$

Proof of (i). We have

$$\sqrt{\sqrt{\sqrt{z-b}-c}-d} = \left(((z-b)^{1/2}-c)^{1/2}-d\right)^{1/2} , \tag{3}$$

hence, operating N^{γ} to the both sides of (3), we obtain

$$\left(\sqrt{\sqrt{\sqrt{z-b}-c}-d}\right)_{\gamma} = \left(\left(((z-b)^{1/2}-c)^{1/2}-d\right)^{1/2}\right)_{\gamma} \tag{4}$$

$$= e^{-i\pi\gamma} (z-b)^{(1/8)-\gamma}$$

$$\times \sum_{m,k=0}^{\infty} \frac{[-\frac{1}{2}]_m [\frac{m}{2}-\frac{1}{4}]_k \Gamma(\frac{k}{2}-\frac{1}{8}+\frac{m}{4}+\gamma)}{m! \cdot k! \ \Gamma(\frac{k}{2}-\frac{1}{8}+\frac{m}{4})} \left(\frac{c}{\sqrt{z-b}}\right)^k \left(\frac{d}{\sqrt[4]{z-b}}\right)^m \tag{5}$$

setting $\alpha = \beta = \delta = 1/2$ in Theorem E. (i) in preliminary, under the conditions stated befor.

Proof of (i i). Set $\gamma = n$ in (i).

Corollary 1. *We have*

(i) $$\left(\sqrt{\sqrt{\sqrt{z-b}-c}}\right)_{\gamma} = e^{-i\pi\gamma} (z-b)^{(1/8)-\gamma}$$

$$\times \sum_{k=0}^{\infty} \frac{[-\frac{1}{4}]_k \Gamma(\frac{k}{2}-\frac{1}{8}+\gamma)}{k! \ \Gamma(\frac{k}{2}-\frac{1}{8})} \left(\frac{c}{\sqrt{z-b}}\right)^k \tag{6}$$

$$\left(\left|\frac{\Gamma(\frac{k}{2}-\frac{1}{8}+\gamma)}{\Gamma(\frac{k}{2}-\frac{1}{8})}\right| < \infty\right)$$

and

(ii) $$\left(\sqrt{\sqrt{\sqrt{z-b}-c}}\right)_n = (-1)^n (z-b)^{(1/8)-n} \times \sum_{k=0}^{\infty} \frac{[-\frac{1}{4}]_k [\frac{k}{2}-\frac{1}{8}]_n}{k!} \left(\frac{c}{\sqrt{z-b}}\right)^k \tag{7}$$

$(n \in \mathbf{Z}_0^+)$

where

$$\left|\frac{c}{\sqrt{z-b}}\right| < 1 \ .$$

Proof of (i). Set $d = 0$ in Theorem 1 (i).

We have (6) from Theorem D, (i), setting $\beta = 1/2$ and $\alpha = 1/4$.

Proof of (ii). Set $\gamma = n$ in (i).

Corollary 2. *We have*

(i) $$\left(\sqrt{\sqrt{\sqrt{z-b}}}\right)_\gamma = e^{-i\pi\gamma} \frac{\Gamma(-\frac{1}{8}+\gamma)}{\Gamma(-\frac{1}{8})} (z-b)^{(1/8)-\gamma} \tag{8}$$

$(|\Gamma(-\frac{1}{8}+\gamma)| < \infty)$

and

(ii) $$\left(\sqrt{\sqrt{\sqrt{z-b}}}\right)_n = (-1)^n [-\tfrac{1}{8}]_n (z-b)^{(1/8)-n} \tag{9}$$

$(n \in \mathbf{Z}_0^+)$

where

$$z - b \neq 0 \ .$$

Proof of (i). Set $c = 0$ in Corollary 1 (i).

Or set $c = d = 0$ in Theorem 1. (i).

Proof of (ii). Set $\gamma = n$ in (i).

Note. We have

$$\left(\sqrt{\sqrt{\sqrt{z-b}}}\right)_\gamma = ((z-b)^{1/8})_\gamma = e^{-i\pi\gamma} \frac{\Gamma(-\frac{1}{8}+\gamma)}{\Gamma(-\frac{1}{8})} (z-b)^{(1/8)-\gamma} \tag{10}$$

from Lemma (i), directly.

§ 3. Semi Derivatives and Integrals

Theorem 2. *We have*

(i) $$\left(\sqrt{\sqrt{\sqrt{z-b}-c}-d}\right)_{1/2} = -i(z-b)^{-3/8}$$

$$\times \sum_{m,k=0}^{\infty} \frac{[-\frac{1}{2}]_m[\frac{m}{2}-\frac{1}{4}]_k\Gamma(\frac{k}{2}+\frac{3}{8}+\frac{m}{4})}{m!\cdot k!\,\Gamma(\frac{k}{2}-\frac{1}{8}+\frac{m}{4})} \left(\frac{c}{\sqrt{z-b}}\right)^k \left(\frac{d}{\sqrt[4]{z-b}}\right)^m \qquad (1)$$

(semi derivatives)

and

(ii) $$\left(\sqrt{\sqrt{\sqrt{z-b}-c}-d}\right)_{-1/2} = i(z-b)^{5/8}$$

$$\times \sum_{m,k=0}^{\infty} \frac{[-\frac{1}{2}]_m[\frac{m}{2}-\frac{1}{4}]_k\Gamma(\frac{k}{2}-\frac{5}{8}+\frac{m}{4})}{m!\cdot k!\,\Gamma(\frac{k}{2}-\frac{1}{8}+\frac{m}{4})} \left(\frac{c}{\sqrt{z-b}}\right)^k \left(\frac{d}{\sqrt[4]{z-b}}\right)^m \qquad (2)$$

(semi integrals)

where

$$\left|\frac{c}{\sqrt{z-b}}\right| < 1 \ , \quad \left|\frac{d}{\sqrt[4]{z-b}}\right| < 1 \ .$$

Proof. Set $\gamma = 1/2$ and $-1/2$ in Theorem 1 (i) we have then (i) and (i i) respectively, clearly.

Corollary 3. *We have*

(i) $$\left(\sqrt{\sqrt{\sqrt{z-b}-c}}\right)_{1/2} = -i(z-b)^{-3/8} \sum_{k=0}^{\infty} \frac{[-\frac{1}{4}]_k\Gamma(\frac{k}{2}+\frac{3}{8})}{k!\,\Gamma(\frac{k}{2}-\frac{1}{8})} \left(\frac{c}{\sqrt{z-b}}\right)^k \qquad (3)$$

(semi derivatives)

and

(ii) $$\left(\sqrt{\sqrt{\sqrt{z-b}-c}}\right)_{-1/2} = i(z-b)^{5/8} \sum_{k=0}^{\infty} \frac{[-\frac{1}{4}]_k\Gamma(\frac{k}{2}-\frac{5}{8})}{k!\,\Gamma(\frac{k}{2}-\frac{1}{8})} \left(\frac{c}{\sqrt{z-b}}\right)^k \qquad (4)$$

(semi integrals)

where

$$\left|\frac{c}{\sqrt{z-b}}\right| < 1 \ .$$

Proof. Set $d = 0$ in Theorem 2.

Corollary 4. *We have*

$$\text{(i)} \quad \left(\sqrt{\sqrt{\sqrt{z-b}-d}}\right)_{1/2} = -i(z-b)^{-3/8} \sum_{m=0}^{\infty} \frac{[-\frac{1}{2}]_m \Gamma(\frac{3}{8}+\frac{m}{4})}{m!! \, \Gamma(-\frac{1}{8}+\frac{m}{4})} \left(\frac{d}{\sqrt[4]{z-b}}\right)^m \qquad (5)$$

(semi derivatives)

and

$$\text{(ii)} \quad \left(\sqrt{\sqrt{\sqrt{z-b}-d}}\right)_{-1/2} = i(z-b)^{5/8} \sum_{m,k=0}^{\infty} \frac{[-\frac{1}{2}]_m \Gamma(-\frac{5}{8}+\frac{m}{4})}{m! \, \Gamma(-\frac{1}{8}+\frac{m}{4})} \left(\frac{d}{\sqrt[4]{z-b}}\right)^m \qquad (2)$$

(semi integrals)

where

$$\left|\frac{d}{\sqrt[4]{z-b}}\right| < 1 \; .$$

Proof. Set $c = 0$ in Theorem 2.

§ 4. Some Special Cases

[I] When $n = 0$, we have

$$\left(\sqrt{\sqrt{\sqrt{z-b}-c}-d}\right)_0 = (z-b)^{1/8}$$

$$\times \sum_{m,k=0}^{\infty} \frac{[-\frac{1}{2}]_m [\frac{m}{2}-\frac{1}{4}]_k}{m! \cdot k!} \left(\frac{c}{\sqrt{z-b}}\right)^k \left(\frac{d}{\sqrt[4]{z-b}}\right)^m \qquad (1)$$

from § 2. (2).

Now we have

$$\text{RHS of (1)} = (z-b)^{1/8} \sum_{m=0}^{\infty} \frac{[-\frac{1}{2}]_m}{m!} T^m \sum_{k=0}^{\infty} \frac{[\frac{m}{2}-\frac{1}{4}]_k}{k!} S^k \qquad (2)$$

$$\left(S = \frac{c}{\sqrt{z-b}}, \quad T = \frac{d}{\sqrt[4]{z-b}} \right)$$

$$= (z-b)^{1/8} \sum_{m=0}^{\infty} \frac{[-\frac{1}{2}]_m}{m!} T^m (1-S)^{(1/4)-(m/2)} \qquad (3)$$

$$= (z-b)^{1/8}(1-S)^{1/4}\sum_{m=0}^{\infty}\frac{[-\frac{1}{2}]_m T^m}{m!\,(1-S)^{m/2}} \tag{4}$$

$$= (z-b)^{1/8}(1-S)^{1/4}\sum_{m=0}^{\infty}\frac{[-\frac{1}{2}]_m}{m!}\left(\frac{d}{\sqrt{\sqrt{z-b}-c}}\right)^m \tag{5}$$

$$= (z-b)^{1/8}\left(1-\frac{c}{\sqrt{z-b}}\right)^{1/4}\left(1-\frac{d}{\sqrt{\sqrt{z-b}-c}}\right)^{1/2} \tag{6}$$

$$= \sqrt{\sqrt{\sqrt{z-b}-c}-d} \qquad (\text{ LHS of } (1)\,)\,. \tag{7}$$

[I I] When $n=1$, we have

$$\left(\sqrt{\sqrt{\sqrt{z-b}-c}-d}\right)_1 = -(z-b)^{-7/8}\sum_{m,k=0}^{\infty}\frac{[-\frac{1}{2}]_m[\frac{m}{2}-\frac{1}{4}]_k[\frac{k}{2}+\frac{m}{4}-\frac{1}{8}]_1}{m!\cdot k!}S^k\,T^m \tag{8}$$

$$= -(z-b)^{-7/8}\sum_{m,k=0}^{\infty}\frac{[-\frac{1}{2}]_m[\frac{m}{2}-\frac{1}{4}]_k(\frac{k}{2}+\frac{m}{4}-\frac{1}{8})}{m!\cdot k!}S^k\,T^m \tag{9}$$

from § 2. (2).

Now we have the identities;

$$\sum_{k=0}^{\infty}\frac{[\lambda]_k}{k!}z^k = (1-z)^{-\lambda}\ , \tag{10}$$

$$[\lambda]_0 = 1, \qquad [\lambda]_1 = \lambda\ , \tag{11}$$

$$[\lambda]_{k+1} = \lambda[\lambda+1]_k\ , \tag{12}$$

hence

$$\sum_{k=0}^{\infty}\frac{[\lambda]_k k}{k!}z^k = \sum_{k=1}^{\infty}\frac{[\lambda]_k}{(k-1)!}z^k = z\sum_{k=0}^{\infty}\frac{[\lambda]_{k+1}}{k!}z^k$$

$$= \lambda z(1-z)^{-\lambda-1}\,. \tag{13}$$

Therefore, we have

$$\sum_{m,k=0}^{\infty}\frac{[-\frac{1}{2}]_m[\frac{m}{2}-\frac{1}{4}]_k(\frac{k}{2})}{m!\cdot k!}S^k\,T^m = \frac{1}{2}\sum_{m=0}^{\infty}\frac{[-\frac{1}{2}]_m}{m!}T^m\sum_{k=0}^{\infty}\frac{[\frac{m}{2}-\frac{1}{4}]_k k}{k!}S^k \tag{14}$$

$$=\frac{1}{2}\sum_{m=0}^{\infty}\frac{[-\frac{1}{2}]_m}{m!}T^m\left(\frac{m}{2}-\frac{1}{4}\right)S\sum_{k=0}^{\infty}\frac{[\frac{m}{2}+\frac{3}{4}]_k}{k!}S^k \tag{15}$$

$$=\frac{1}{2}S(1-S)^{-3/4}\sum_{m=0}^{\infty}\frac{[-\frac{1}{2}]_m(\frac{m}{2}-\frac{1}{4})}{m!}U^m \qquad \left(U=\frac{d}{\sqrt{\sqrt{z-b}-c}}\right) \tag{16}$$

$$=\frac{1}{2}S(1-S)^{-3/4}\left\{-\frac{1}{4}U(1-U)^{-1/2}-\frac{1}{4}(1-U)^{1/2}\right\} \tag{17}$$

$$=-\frac{1}{8}S(1-S)^{-3/4}(1-U)^{-1/2}\ , \tag{18}$$

$$\sum_{m,k=0}^{\infty}\frac{[-\frac{1}{2}]_m[\frac{m}{2}-\frac{1}{4}]_k(\frac{m}{4})}{m!\cdot k!}S^kT^m=\frac{1}{4}\sum_{m=0}^{\infty}\frac{[-\frac{1}{2}]_m m}{m!}T^m\sum_{k=0}^{\infty}\frac{[\frac{m}{2}-\frac{1}{4}]_k}{k!}S^k \tag{19}$$

$$=\frac{1}{4}(1-S)^{1/4}\sum_{m=1}^{\infty}\frac{[-\frac{1}{2}]_m}{(m-1)!}U^m \tag{20}$$

$$=\frac{1}{4}(1-S)^{1/4}U\left(-\frac{1}{2}\right)\sum_{m=0}^{\infty}\frac{[\frac{1}{2}]_m}{m!}U^m \tag{21}$$

$$=-\frac{1}{8}(1-S)^{1/4}U(1-U)^{-1/2}\ . \tag{22}$$

and

$$\sum_{m,k=0}^{\infty}\frac{[-\frac{1}{2}]_m[\frac{m}{2}-\frac{1}{4}]_k(-\frac{1}{8})}{m!\cdot k!}S^kT^m=-\frac{1}{8}\sum_{m=0}^{\infty}\frac{[-\frac{1}{2}]_m}{m!}T^m\sum_{k=0}^{\infty}\frac{[\frac{m}{2}-\frac{1}{4}]_k}{k!}S^k \tag{23}$$

$$=-\frac{1}{8}(1-S)^{1/4}\sum_{m=0}^{\infty}\frac{[-\frac{1}{2}]_m}{m!}U^m \tag{24}$$

$$=-\frac{1}{8}(1-S)^{1/4}(1-U)^{1/2}\ . \tag{25}$$

Then applying (18), (22) and (25) to (9) we obtain

$$\left(\sqrt{\sqrt{\sqrt{z-b}-c}-d}\right)_1=\frac{1}{8}(z-b)^{-7/8}(1-S)^{1/4}(1-U)^{-1/2}\{S(1-S)^{-1}+1\} \tag{26}$$

$$=\frac{1}{8}(z-b)^{-7/8}(1-S)^{1/4}(1-U)^{-1/2}(1-S)^{-1} \qquad (27)$$

$$=\frac{1}{8}(z-b)^{-7/8}(1-S)^{-3/4}(1-U)^{-1/2} \qquad (28)$$

$$=\frac{1}{8}(z-b)^{-1/2}(\sqrt{z-b}-c)^{-1/2}(\sqrt{\sqrt{z-b}-c}-d)^{-1/2} \quad . \qquad (29)$$

This result (29) coincides with the one obtained by classical calculus.

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